$$\frac{\partial X_i}{\partial x_j} = \delta_{ij} - am_i n_j \qquad [14c]$$

If we take a Cartesian coordinate system with the normal **n** to the slip plane as axis 1 and the slip direction **m** as axis 2, the components of **n** are (1, 0, 0) and those of **m** are (0, 1, 0). Matrix [14] then becomes

$$F = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [15]$$

With the deformation gradient matrix given by Eq. [15], Eq. [5] reduces to

$$\lambda_{\mathbf{P}}^2 = \left(\frac{l_1}{l_0}\right)^2 = 1 + a^2P_1^2 + 2aP_1P_2 \qquad [16]$$

while Eq. [6] yields

$$p_1 = \frac{l_0}{l_1} P_1 \qquad [17a]$$

$$p_2 = \frac{l_0}{l_1}(aP_1 + P_2) \qquad [17b]$$

$$p_3 = \frac{l_0}{l_1} P_3 \qquad [17c]$$

Here $P_1$, $P_2$, and $P_3$ are the direction cosines (with respect to the same coordinates as **m** and **n** above) of the initial direction of any arbitrary material line; $p_1$, $p_2$, and $p_3$ are the corresponding values after the deformation; $l_1/l_0$ is the ratio of final to initial length.

These formulas are applicable to tensile testing when the deformation corresponds to a single active slip system. The grip system maintains the direction of the material line along the tensile axis. This line, however, rotates with respect to the lattice, and hence with respect to our coordinate system which is fixed in the lattice. With **P** along the tensile axis, the above formulas enable one to find the length ratio $l_1/l_0$ and the rotation of the tensile axis with respect to the lattice. The amount of shear, $a$, can be expressed in terms of the initial and final positions of the tensile axis by solving Eq. [17b] for $a$ after substituting for $l_0/l_1$ from [17a]. The result is

$$a = \frac{p_2}{p_1} - \frac{P_2}{P_1} \qquad [18]$$

Eqs. [16] to [18] have been derived previously by Mark, Polanyi, and Schmid.[3]

As a specific application, Fig. 1 shows a standard (001) stereographic projection. If the tensile axis $P$ of a single-crystal rod lies anywhere within the standard [001]–[$\bar{1}$11]–[011] triangle, then according to the Schmid law the active slip system for a fcc crystal is (111)[$\bar{1}$01] (the primary slip system). It is convenient to use [111], [$\bar{1}$01], [1$\bar{2}$1] as Cartesian coordinate axes, in which case the deformation gradient matrix is given by Eq. [15], and the remaining formulas [16] to [18] are directly applicable.

## TWO OR MORE SLIP SYSTEMS

In extending the treatment to two (or more) slip systems $A$ and $B$, we express the corresponding deformation gradient matrices (see Eq. [14a]) as

$$F_A = I + am_A n_A^T$$
$$F_B = I + bm_B n_B^T = I + \beta am_B n_B^T \qquad [19]$$

where $\beta = b/a$ is the ratio of glide-shear of the two slip systems. If shear in $A$ is followed by shear in $B$, the deformation gradient matrix for the combination is

$$F_BF_A = I + a(m_An_A^T + \beta m_Bn_B^T) + a^2\beta(m_Bn_B^Tm_An_A^T)$$
$$= I + aF_1 + a^2F_2 \qquad [20]$$

where $F_1 = m_An_A^T + \beta m_Bn_B^T$ and $F_2 = \beta m_Bn_B^Tm_An_A^T$. If shear in $B$ is followed by shear in $A$, the combined result, given by $F_AF_B$, is the same except that $F_2 = \beta m_An_A^Tm_Bn_B^T$. Note in general $m_An_A^Tm_Bn_B^T \neq m_Bn_B^Tm_An_A^T$ since this is a matrix product.

Physically, we imagine that the final configuration resulting from the operation of the two slip systems is reached by a long series of steps in which a *small* deformation $F_A$ (or $F_B$) is followed by a *small* deformation $F_B$ (or $F_A$). Thus, we expect to represent the final configuration mathematically by a deformation gradient matrix which is the limit of $(F_BF_A)^N$ as $N \to \infty$ while $a \to 0$ in such a way that the product $Na = \alpha$, a finite constant designating the accumulated amount of shear in slip system $A$. The desired limit is

$$F = \lim_{a\to 0} (F_BF_A)^{\alpha/a} = \lim_{a\to 0} (I + aF_1 + a^2F_2)^{\alpha/a}$$
$$= I + \alpha F_1 + \frac{1}{2}\alpha^2F_1^2 + \frac{1}{3!}\alpha^3F_1^3 + \cdots$$
$$= e^{\alpha F_1} \qquad [21]$$

Since $F_2$ does not enter the final result, $(F_AF_B)^N$ has the same limit. Thus, as expected, the final configuration is independent of the exact sequence of opera-

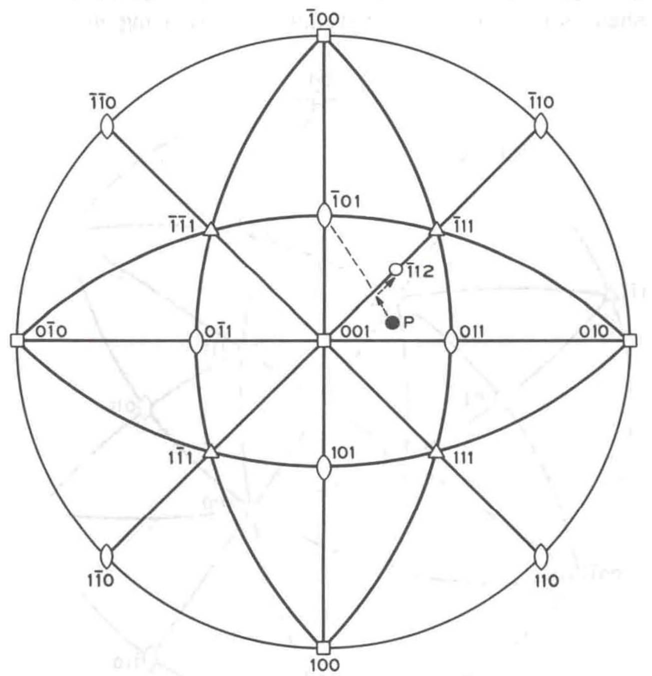


Fig. 1—Standard (001) stereographic projection for a cubic crystal. A single crystal whose tensile axis $P$ lies inside the triangle deforms by (111)[$\bar{1}$01] (primary) slip. If tensile axis lies along the [001]-[$\bar{1}$11] line, equal slip on (111)[$\bar{1}$01] (primary) and ($\bar{1}\bar{1}$1)[011] (conjugate) results. Arrows indicate path of axial rotation.

tion of the two slip systems, in contrast to the result of a sequence of two *finite* shears. It can easily be shown that Eq. [21] is likewise applicable to more than two slip systems. The matrix $F_1$ will in general take the form $F_1 = m_A n_A^T + \beta m_B n_B^T + \gamma m_C n_C^T + \cdots$.

It remains now to evaluate the matrix $e^{\alpha F_1}$ of Eq. [21]. We first note that any similarity transformation that diagonalizes $\alpha F_1$ also diagonalizes $e^{\alpha F_1}$. Let us suppose that a nonsingular matrix $S$ has been found such that

$$S(\alpha F_1)S^{-1} = \alpha \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \equiv \alpha\Lambda \qquad [22]$$

Then,

$$e^{\alpha\Lambda} = S(e^{\alpha F_1})S^{-1} = \begin{bmatrix} e^{\alpha\lambda_1} & 0 & 0 \\ 0 & e^{\alpha\lambda_2} & 0 \\ 0 & 0 & e^{\alpha\lambda_3} \end{bmatrix} \qquad [23]$$

and $e^{\alpha F_1}$ can be found from

$$e^{\alpha F_1} = S^{-1}[S(e^{\alpha F_1})S^{-1}]S = S^{-1}(e^{\alpha\Lambda})S \qquad [24]$$

This evaluation requires diagonalization of the matrix. E. N. Gilbert[9] has shown us an elegant method, presented in the Appendix, of evaluating $e^{\alpha F_1}$ without diagonalization. The calculations will now be illustrated with the following deformation.

## CASE OF (110)[$\bar{1}$12] COMPRESSION

Fig. 2 shows the standard (110) stereographic projection. If a fcc single crystal is compressed on the (110) plane and constrained to flow in the [$\bar{1}$12] direction (by confining the crystal to a channel), slip will occur equally in the two systems $A \equiv (111)[10\bar{1}]$ and $B \equiv (11\bar{1})[011]$ as a result of a favorable resolved shear stress on these systems. In evaluating $m_A$, $m_B$,

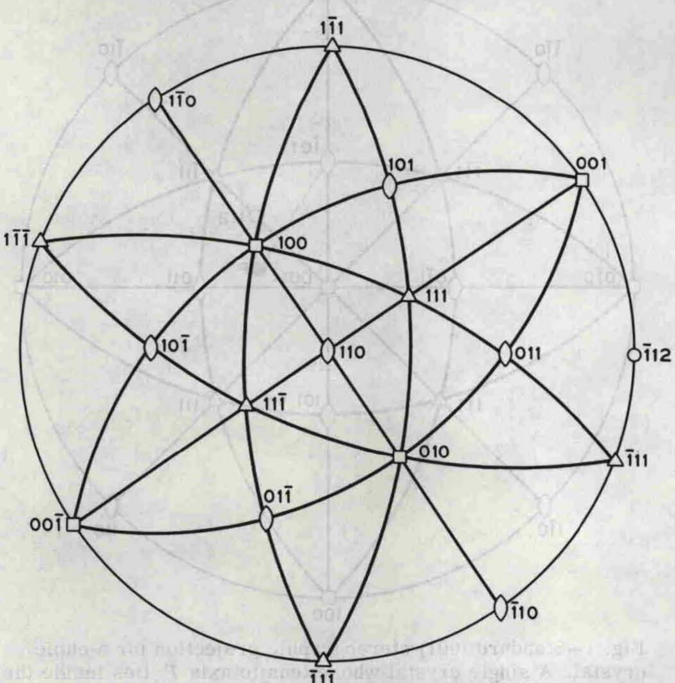


Fig. 2—Standard (110) stereographic projection. For compression on (110) and elongation in [$\bar{1}$12], the active slip systems are $(11\bar{1})[011]$ and $(111)[10\bar{1}]$.

and so forth, of Eq. [20], it is convenient to take the specimen axes as Cartesian coordinates, *i.e.*, let $X_1, X_2, X_3$ be respectively along [110], [$\bar{1}1\bar{1}$], [$\bar{1}$12], Fig. 2. The matrix of transformation from cubic axes to those above is

| | [100] | [010] | [001] |
|---|---|---|---|
| [110] | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 0 |
| [$\bar{1}1\bar{1}$] | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{\sqrt{3}}$ |
| [$\bar{1}$12] | $-\frac{1}{\sqrt{6}}$ | $\frac{1}{\sqrt{6}}$ | $\frac{2}{\sqrt{6}}$ |

[25]

Hence, if $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are unit vectors along the specimen axes and $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3$ along the cubic axes, we have, for $(111)[10\bar{1}]$ slip,

$$\mathbf{n}_A = \frac{1}{\sqrt{3}}(\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3) = \frac{2}{\sqrt{6}}\mathbf{i}_1 - \frac{1}{3}\mathbf{i}_2 + \frac{\sqrt{2}}{3}\mathbf{i}_3$$

$$\mathbf{m}_A = \frac{1}{\sqrt{2}}(\mathbf{I}_1 - \mathbf{I}_3) = \frac{1}{2}\mathbf{i}_1 - \frac{\sqrt{3}}{2}\mathbf{i}_3 \qquad [26]$$

and for $(11\bar{1})[011]$ slip,

$$\mathbf{n}_B = \frac{2}{\sqrt{3}}(\mathbf{I}_1 + \mathbf{I}_2 - \mathbf{I}_3) = \frac{2}{\sqrt{6}}\mathbf{i}_1 + \frac{1}{3}\mathbf{i}_2 - \frac{\sqrt{2}}{3}\mathbf{i}_3$$

$$\mathbf{m}_B = \frac{1}{\sqrt{2}}(\mathbf{I}_2 + \mathbf{I}_3) = \frac{1}{2}\mathbf{i}_1 + \frac{\sqrt{3}}{2}\mathbf{i}_3 \qquad [27]$$

From Eqs. [14] and [26], we have

$$F_A = \begin{bmatrix} 1 - \frac{a}{\sqrt{6}} & \frac{a}{6} & -\frac{a}{3\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{a}{\sqrt{2}} & -\frac{a}{2\sqrt{3}} & 1 + \frac{a}{\sqrt{6}} \end{bmatrix} \qquad [28]$$

where shear in the negative sense has been chosen to conform with compression along $X_1$.

Similarly, Eqs. [14] and [27] yield

$$F_B = \begin{bmatrix} 1 - \frac{b}{\sqrt{6}} & -\frac{b}{6} & \frac{b}{3\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{b}{\sqrt{2}} & -\frac{b}{2\sqrt{3}} & 1 + \frac{b}{\sqrt{6}} \end{bmatrix} \qquad [29]$$

Hence in the expected case of equal slip, $b = a$,

$$F_B F_A = \begin{bmatrix} 1 - \frac{2a}{\sqrt{6}} + \frac{a^2}{3} & -\frac{a^2}{3\sqrt{6}} & \frac{a^2}{3\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{a^2}{\sqrt{3}} & -\frac{a}{\sqrt{3}} - \frac{a^2}{3\sqrt{2}} & 1 + \frac{2a}{\sqrt{6}} + \frac{a^2}{3} \end{bmatrix} \qquad [30]$$

In the form of Eq. [20], $F_B F_A = I + aF_1 + a^2 F_2$,

$$F_1 = \begin{bmatrix} -\frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$